Stage 2 Specialist Mathematics

Complex Numbers and Real Polynomials Test

Topic 2: Subtopics 2.1, 2.2, 2.3, 2.4

Total Marks - 47

(Calculator and one A4 page of handwritten notes permitted.)

Question 1 (5 marks)

Let $f(x) = x^3 + ax + b$, where a and b are constants.

(a) Given that $(x + 1)$ is a factor of $f(x)$, show that $b - a = 1$.

$f(-1) = 0$ [factor theorem]

$\therefore -1 - a + b = 0$

$\therefore b - a = 1$ ①

(2 marks)

(b) When $f(x)$ is divided by $(x + 2)$, the remainder is 6.

Show that $b - 2a = 14$.

$f(-2) = 6$ [remainder theorem]

$\therefore -8 - 2a + b = 6$

$\therefore b - 2a = 14$ ②

(1 mark)

(c) Find a and b, and hence write $f(x)$ as a product of linear factors.

① − ② $\Rightarrow$ $a = -13$

Now ① $\Rightarrow$ $b = -12$

$$\begin{array}{r|rrrr} -1 & 1 & 0 & -13 & -12 \\ & 0 & -1 & 1 & 12 \\ \hline & 1 & -1 & -12 & 0 \end{array}$$

[synthetic division to divide $f(x)$ by $(x+1)$]

$\therefore f(x) = (x+1)(x^2 - x - 12)$

$= (x+1)(x+3)(x-4)$

(2 marks)

Question 2 (10 marks)

(a) Let $f(x) = x^3 + 7$.

Show that $f(x) - f(2)$ has a factor of $x - 2$.

Let $F(x) = f(x) - f(2)$

$= x^3 - 8$

$F(2) = 0$

$\therefore (x-2)$ is a factor of $f(x) - f(2)$

(2 marks)

(b) If $p(x)$ is any polynomial of degree ≥ 1, prove that $p(x) - p(k)$ has a factor of $x - k$.

Let $P(x) = p(x) - p(k)$

$P(k) = 0$

$\therefore (x-k)$ is a factor of $p(x) - p(k)$

(2 marks)

(c) $T(x)$ is a real cubic polynomial with a zero of $1 + 2i$.

(i) Find a real quadratic factor of $T(x)$.

$\alpha = 1 + 2i$ is a zero of $T(x)$

$\therefore \beta = 1 - 2i$ is a zero of $T(x)$

$\alpha + \beta = 2$

$\alpha\beta = 5$

$\therefore x^2 - 2x + 5$ is a real quadratic factor of $T(x)$

(3 marks)

(ii) Find $T(x)$, given that $T(x) - 25$ has a factor of $x - 2$ and that $T(x) - 12$ has a factor of $x - 1$.

$T(x) = (ax + b)(x^2 - 2x + 5)$

$T(2) - 25 = 0$ [factor theorem]

$\therefore 5(2a + b) = 25$

$\therefore 2a + b = 5$ ①

$T(1) - 12 = 0$ [factor theorem]

$\therefore 4(a + b) = 12$

$\therefore a + b = 3$ ②

① - ② $\Rightarrow$ $a = 2$

Now ② $\Rightarrow$ $b = 1$

$\therefore T(x) = (2x + 1)(x^2 - 2x + 5)$

(3 marks)

Question 3 (9 marks)

Let $p(z) = z^6 + az^4 + bz^2 + c$, where a, b, and c are real numbers.

(a) (i) Given that $z = 1 + i$ is a zero of $p(z)$, show that $z^2 - 2z + 2$ is a quadratic factor of $p(z)$.

$\alpha = 1 + i$ is a zero of $p(z)$

$\therefore \beta = 1 - i$ is a zero of $p(z)$

$\alpha + \beta = 2$

$\alpha\beta = 2$

$\therefore z^2 - 2z + 2$ is a quadratic factor of $p(z)$

(2 marks)

(ii) If $z = -1 + i$ is also a zero of $p(z)$, show that $z^4 + 4$ is a factor of degree 4 of $p(z)$.

$\alpha = -1 + i$ is a zero of $p(z)$

$\therefore \beta = -1 - i$ is a zero of $p(z)$

$\alpha + \beta = -2$

$\alpha\beta = 2$

$\therefore z^2 + 2z + 2$ is a quadratic factor of $p(z)$

$(z^2 - 2z + 2)(z^2 + 2z + 2) = z^4 + \cancel{2z^3} + \cancel{2z^2} - \cancel{2z^3} - \cancel{4z^2} - \cancel{4z} + \cancel{2z^2} + \cancel{4z} + 4$

$= z^2 + 4$

$\therefore z^4 + 4$ is a factor of degree 4 of $p(z)$

(2 marks)

(b) (i) Show that $p(z) = (z^4 + 4)(z^2 + a) + (b - 4)z^2 + c - 4a$.

$(z^4 + 4)(z^2 + a) + (b - 4)z^2 + c - 4a = z^6 + az^4 + \cancel{4z^2} + \cancel{4a} + bz^2 - \cancel{4z^2} + c - \cancel{4a}$

$= z^6 + az^4 + bz^2 + c$

$= p(z)$

(1 mark)

(ii) Given that $z^4 + 4$ is a factor of $p(z)$, explain why $b = 4$ and $c = 4a$.

If $z^4 + 4$ is a factor of $p(z)$ then the remainder when $p(z)$ is divided by $z^4 + 4$ is zero

$(b-4)z^2 + c - 4a$ is the remainder when $p(z)$ is divided by $z^4 + 4$

$\therefore b - 4 = 0$ and $c - 4a = 0$

i.e. $b = 4$ and $c = 4a$

(1 mark)

(c) (i) If the remainder is 60 when $p(z)$ is divided by $(z-2)$, show that $16a + 4b + c = -4$.

$p(2) = 60$ [remainder theorem]

$\therefore 64 + 16a + 4b + c = 60$

ie. $16a + 4b + c = -4$ ①

(2 marks)

(ii) Using the information from parts (b) (ii) and (c) (i), solve for a and c, and hence find $p(z)$.

Substituting $b=4$ and $c=4a$ into ① gives $16a + 16 + 4a = -4$

$20a = -20$

$a = -1$

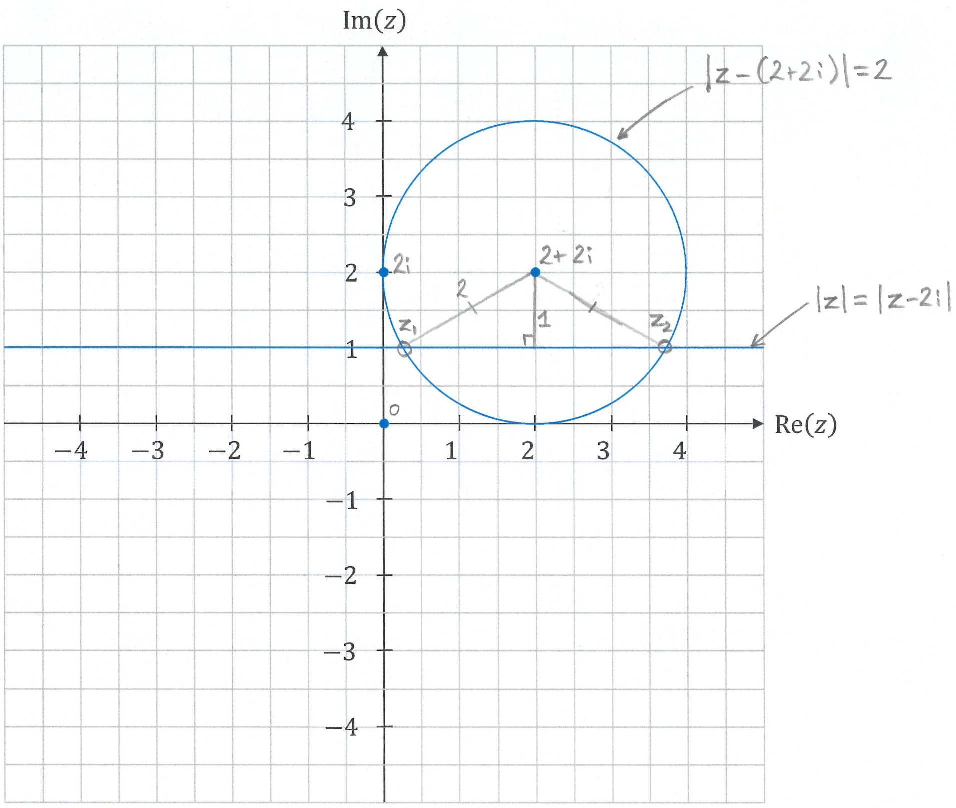
$\therefore p(z) = z^6 - z^4 + 4z^2 - 4$

(1 mark)

Question 4 (8 marks)

(a) On the Argand diagram below, show the set of complex numbers such that

$$|z - (2 + 2i)| = 2$$



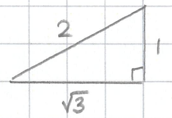
(2 marks)

(b) On the same Argand diagram, show the set of complex numbers such that

$$|z| = |z - 2i|$$

(2 marks)

(c) Find in Cartesian form the exact value of all complex numbers that satisfy *both* the equation in part (a) *and* the equation in part (b).



$z_1 = 2 + 2i - \sqrt{3} - i$

$= (2 - \sqrt{3}) + i$

$z_2 = 2 + 2i + \sqrt{3} - i$

$= (2 + \sqrt{3}) + i$

(4 marks)

Question 5 (15 marks)

(a) (i) Write $\sqrt{2} - i\sqrt{2}$ exactly in $r \operatorname{cis} \theta$ form, where $r \geq 0$ and $-\pi < \theta \leq \pi$.

$2 \operatorname{cis}\left(\frac{-\pi}{4}\right)$

(2 marks)

(ii) Hence find $(\sqrt{2} - i\sqrt{2})^4$.

$16 \operatorname{cis}(-\pi) = -16$

(1 mark)

(b) Solve $z^4 = -16$, writing your answers exactly in $r \operatorname{cis} \theta$ form.

$z^4 = 16 \operatorname{cis}(\pi + k.2\pi)$ where $k \in \mathbb{Z}$

$\therefore z = 2 \operatorname{cis}(\pi/4 + k.2\pi/4)$

$z = 2\operatorname{cis}\left(\frac{\pi}{4}\right), 2\operatorname{cis}\left(\frac{3\pi}{4}\right), 2\operatorname{cis}\left(-\frac{3\pi}{4}\right), 2\operatorname{cis}\left(-\frac{\pi}{4}\right)$

(3 marks)

(c) Show that $\frac{z^7+z^4+16z^3+16}{z^3+1} = z^4 + 16$.

$\frac{z^7 + z^4 + 16z^3 + 16}{z^3+1} = \frac{z^4(z^3+1) + 16(z^3+1)}{z^3+1}$

$= z^4 + 16$

(1 mark)

(d) Use your results from parts (b) and (c) to solve the equation

$$z^7 + z^4 + 16z^3 + 16 = 0.$$

Write your answers exactly in $r \operatorname{cis} \theta$ form.

$z^7 + z^4 + 16z^3 + 16 = 0 \Rightarrow (z^4 + 16)(z^3 + 1) = 0$

$z^4 + 16 = 0 \Rightarrow z^4 = -16$ (see solutions given in (b))

$z^3 + 1 = 0 \Rightarrow z^3 = -1$

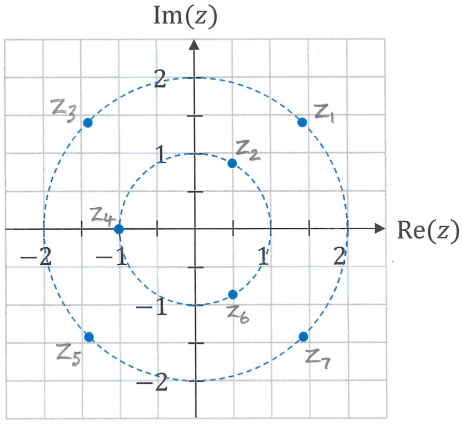
$z^3 = \operatorname{cis}(\pi + k.2\pi)$ where $k \in \mathbb{Z}$

$\therefore z = \operatorname{cis}(\pi/3 + k.2\pi/3)$

$z = \operatorname{cis}\left(\frac{\pi}{3}\right), \operatorname{cis}(\pi), \operatorname{cis}\left(\frac{-\pi}{3}\right)$

(3 marks)

(e) Plot your solutions from part (d) on the Argand diagram below, labelling them $z_1, z_2, \ldots, z_7$ anticlockwise from the smallest positive argument.



(2 marks)

(f) (i) Find $|z_1| + |z_2| + |z_3| + \cdots + |z_7|$.

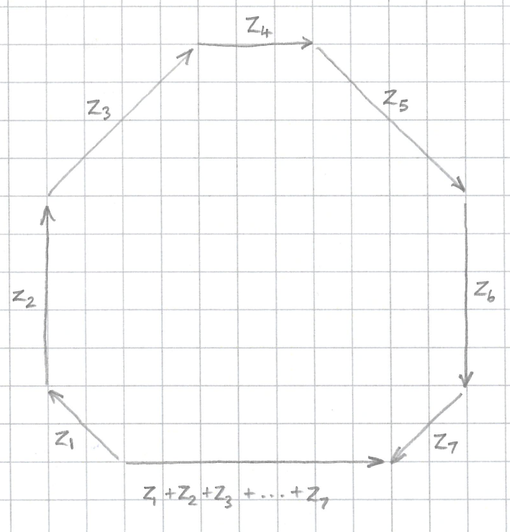
$2 + 1 + 2 + 1 + 2 + 1 + 2$

$= 11$

(1 mark)

(ii) Explain why $|z_1| + |z_2| + |z_3| + \cdots + |z_7| \geq |z_1 + z_2 + z_3 + \cdots + z_7|$.

Consider the octagon illustrated below:



Each side of the octagon must be less than or equal to the sum of the other seven sides.

$\therefore |z_1| + |z_2| + |z_3| + \ldots + |z_7| \geqslant |z_1 + z_2 + z_3 + \ldots + z_7|$

(1 mark)

(iii) Find $|z_1 + z_2 + z_3 + \cdots + z_7|$.

0

(1 mark)