Question 6 (14 marks)

The rate at which medical samples arrive at a laboratory t hours after 8.00 am on a particular day can be modelled using the function

$$m(t) = 500t\sqrt{t}\,e^{-0.9t} \text{ for } 0 \le t \le 9,$$

where the rate at which medical samples arrive at the laboratory, $m(t)$, is measured in samples per hour.

Source: adapted from Belova59 | pixabay.com

(a) (i) Show that $m'(t) = 750\sqrt{t}\,e^{-0.9t} - 450t\sqrt{t}\,e^{-0.9t}$.

$m(t) = 500.t^{3/2}.e^{-0.9t}$

$m'(t) = 500.\frac{3}{2}t^{1/2}.e^{-0.9t} + 500.t^{3/2}.-0.9e^{-0.9t}$

$= 750\sqrt{t}.e^{-0.9t} - 450t\sqrt{t}.e^{-0.9t}$

(2 marks)

(ii) Hence, using an algebraic approach, show that the rate at which medical samples arrive at the laboratory is maximised at $t = \frac{5}{3}$, according to the model.

$m'(t) = 0 \Rightarrow 150\sqrt{t}.e^{-0.9t}(5 - 3t) = 0$

$\therefore t = 0$ or $t = \frac{5}{3}$

$\nearrow \quad \frown \quad \searrow$

sign of $m'(t)$: $+$ on $(0, 5/3)$, $-$ on $(5/3, 9)$; number line: 0, $5/3$, 9, t

$m(t)$ is maximised when $t = \frac{5}{3}$

(3 marks)

Let the total number of medical samples that arrive at the laboratory between 8.00 am and 5.00 pm on a particular day be N.

(b) An overestimate for the value of N can be calculated using three rectangles of equal width. These three rectangles, along with the graph of $y = m(t)$ where $0 \le t \le 9$, are shown in Figure 4.

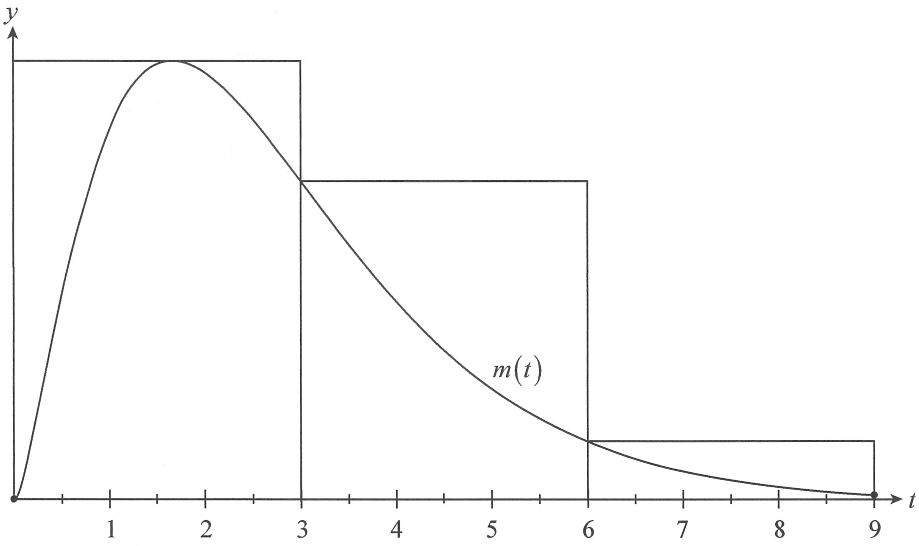

Figure 4

(i) Calculate the value of this overestimate.

Overestimate $= 3 \times m\left(\frac{5}{3}\right) + 3 \times m(3) + 3 \times m(6)$

$= 1340$ units^2 (3s.f.)

(2 marks)

(ii) If an overestimate of N was calculated using an increasing number of rectangles of equal width, it would approach the true value of N.

Calculate this value of N, correct to the nearest integer.

Using technology, $N = \int_0^9 m(t)\,dt$

$= 860$

(1 mark)

When a medical sample arrives at the laboratory, it is placed in a queue before being processed. The medical samples are processed at a constant rate of 150 samples per hour.

Let the function $p(t) = 150$ represent the rate at which medical samples are processed.

Figure 5 shows the graphs of $y = m(t)$ and $y = p(t)$, where $0 \le t \le 9$. The graphs of $y = m(t)$ and $y = p(t)$ intersect at $t = 0.670$ and $t = 3.36$ (correct to three significant figures).

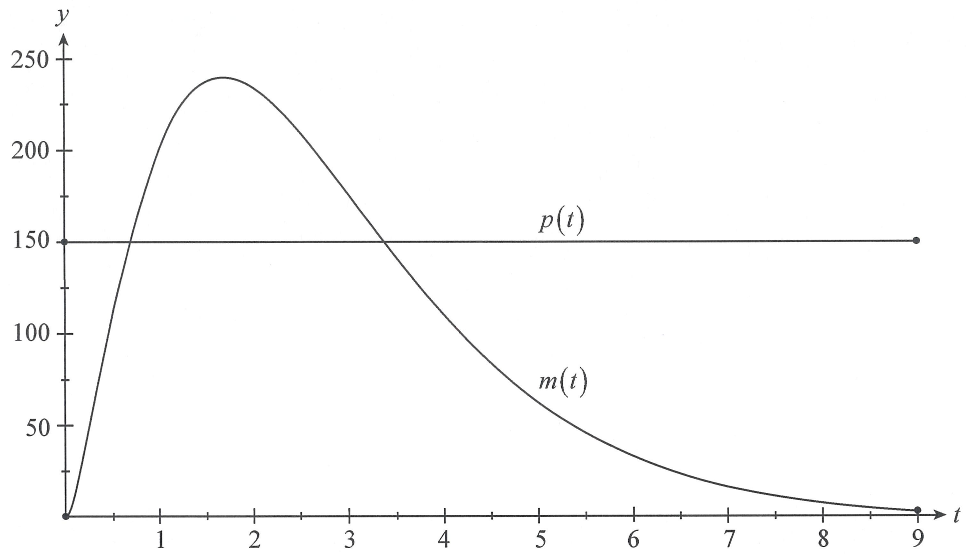

Figure 5

(c) At 8.00 am $(t = 0)$, the laboratory had 600 medical tests in the queue to be processed.

(i) Explain why the number of items in the queue is decreasing until 0.670 hours after 8.00 am.

$m(t) < p(t)$ for $0 \le t < 0.670$

$\therefore$ The number of items in the queue is decreasing during this period

(1 mark)

(ii) During the time that the number of items in the queue was increasing, the queue increased by K medical samples.

(1) Write an integral expression that could be used to calculate K.

$K = \int_{0.670}^{3.36} m(t) - p(t) \, dt$

(2 marks)

(2) Hence, or otherwise, determine the value of K, correct to the nearest integer.

Using technology, $K = 152$

(1 mark)

(iii) Determine the number of medical samples in the queue at 5.00 pm.

Using technology, no. of samples $= 600 + \int_0^9 m(t) - 150 \, dt$

$= 110$

(2 marks)